DIAS-STP 95-32
hep-th/9509125
September 1995

Canonical Quantization of Interacting WZW Theories

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Abstract

Using canonical quantization we find the Virasoro centre for a class of conformally-invariant interacting Wess-Zumino-Witten theories. The theories have a group structure similar to that of Toda theories (both abelian and non-abelian) but the usual Toda constraints on the coupling constants are relaxed and the theories are not necessarily integrable. The general formula for the Virasoro centre is compared to that derived by BRST methods in the Toda case, and helps to explain the structure of the latter.

1. Introduction

In this paper we consider the canonical quantization of a class of conformally-invariant interacting WZW theories. We shall call these theories Toda-like interacting WZW theories as the prototypes are the (abelian and non-abelian) Toda theories and the more general ones obtained are from these by relaxing the constraints on the coupling constants (see below). While the theories so obtained are conformally invariant because of the exponential form of the potential, there is no reason to believe that they are integrable. For example, consider the (abelian) Toda Lagrangians

$$\mathcal{L} = \kappa_{ab}\partial_\mu\phi^a\partial^\mu\phi^b - \sum_a \mu_a^2 e^{\phi^a}, \tag{1.1}$$

where the inverse of κ_{ab} is proportional to $\langle\alpha_a,\alpha_b\rangle$, and the α_a are the simple roots of a simple Lie algebra G. Now if we let κ_{ab} be an arbitrary (symmetric positive definite) matrix we lose integrabilty, but we still have a conformal field theory.

We wish to consider the canonical quantization of Toda-like theories for a number of reasons. First they provide an example of interacting systems, which, although still two-dimensional and conformally invariant, are more general than the systems previously considered. Second, because the quantization is canonical, it provides a useful alternative to the BRST quantization in the Toda case. The structure of the Virasoro central charge for the Toda-like theories explains the structure obtained previously [1,2,3,4] in the Toda case. Finally, because the Toda-like theories are not first-class constrained versions of free theories, the BRST method used [3,5] for the Toda theories themselves is no longer applicable.

For simplicity, we shall actually consider here only Toda-like theories for $sl(N)$ Toda theories with integral $sl(2)$ embeddings [3,6]. This includes the abelian $sl(N)$ case, which corresponds to the principal $sl(2)$ embedding, which is integral. However, it will be clear from the discussion that the procedure generalizes from $sl(N)$ to any simple Lie group and it may well generalize to half-integral embeddings, though we have not investigated this question.

To describe the $sl(N)$ Toda-like theories we first recall that the integral $sl(N)$ Toda theories are defined by Lagrangians of the form

$$\mathcal{L} = \kappa \mathcal{L}^W(g_o) - 2\mathrm{tr}\big(g_o M_+ g_o^{-1} M_-\big) = \kappa \left[\sum_\alpha \mathcal{L}^W(g_\alpha) + \mathcal{L}^W(g_c)\right] - 2\mathrm{tr}\big(g_o M_+ g_o^{-1} M_-\big), \tag{1.2}$$

where $\{M_o, M_\pm\}$ are the standard generators of an embedded $sl(2)$, the (reducible) algebra $G_o = G_c \oplus \sum_\alpha \oplus G_\alpha$ is the little algebra of M_o in $sl(N)$, with centre G_c, and the $\mathcal{L}^W$ are free WZW Lagrangians (see section 2) for the subalgebras indicated. The simple subalgebras G_α are all $sl(n_\alpha)$ subalgebras, where $\sum_\alpha n_\alpha = N$.

Against this background the $sl(N)$ Toda-like theories are described by the Lagrangians

$$\mathcal{L} = \sum_\alpha \kappa_\alpha \mathcal{L}^W(g_\alpha) + \sum_{ab} \kappa_{ab} \partial\phi^a \partial\phi^b - 2\mathrm{tr}\big(g_o N_+ g_o^{-1} N_-\big), \tag{1.3}$$

where the κ_α and κ_{ab} are arbitrary coupling constants, the $N_\pm$ are any constant matrices such that $[M_o, N_\pm] = \pm N_\pm$. This means that the $N_\pm$ intertwine only neighbouring g_α's and thus can be expanded in the form $N_\pm = \sum_a \oplus N_\pm^{(a)}$ where $N_\pm^{(a)}$ intertwines g_α and $g_{\alpha+1}^{-1}$ for $\alpha = a$. Throughout we shall assume the correlation $a \sim (\alpha, \alpha+1)$. The weak non-degeneracy condition $N_\pm^{(a)} \neq 0$ for any a, is also assumed.

We see therefore that the Toda-like theories are obtained from the Toda theories by keeping the little subalgebra G_o fixed but relaxing the conditions on the coupling constants i.e. not requiring that the coefficients κ of the kinetic terms be uniform or that the $N_\pm$ be generators of the embedded $sl(2)$. We shall see that the relaxation of the coupling constants is the important one. Indeed the relaxation $M_\pm \to N_\pm$ plays a role only in so far as the $N_\pm$ are required to be non-degenerate.

What we mean by quantization of these theories is showing that, in the Hilbert space of the quantum Kac-Moody current $J(x)$ of the free WZW theory, a potential $V\big(g(x)\big)$ can be defined in a meaningful way, and there exists a Virasoro operator of the usual form

$$L(x) = T(x) + V\big(g(x)\big) + I(x), \tag{1.4}$$

where $T(x)$ is the Virasoro for the free WZW theory with (reducible) algebra G_o and $I(x)$ is an improvement term of the form $\mathrm{tr}\big(\mathcal{I}\, J'(x)\big)$, where $\mathcal{I}$ is a constant matrix. As in previous models, the improvement term $I(x)$ is introduced because the closure of the Virasoro algebra requires that the potential $V\big(g(x)\big)$ have conformal weights unity. However, with respect to $T(x)+V(x)$, it has zero weight classically and the components V_a in $V = \sum V_a$ acquire anomalous conformal weights $\hbar\lambda_a$ in the quantum theory. The improvement term

is chosen so that $V\big(g(x)\big)$ has conformal weight one with respect to $L(x)$. We show that for the Toda-like theories a suitable improvement term always exists, and that the matrix $\mathcal{I}$ takes the form $\mathcal{I} = M_o - \hbar M_q$, where M_o and $\hbar M_q$ compensate for the classical and anomalous discrepancies $\hbar\lambda_a$ in the conformal weight, respectively. We shall compute both λ_a and M_q explicitly.

We shall show that for Toda-like potentials, the Virasoro centre takes the form

$$C = C_{free} + 24\pi \sum_{ab} \kappa_{ab} \mathrm{tr}\big(\sigma^a \mathcal{I}\big) \mathrm{tr}\big(\sigma^b \mathcal{I}\big), \tag{1.5}$$

where C_{free} is the pure WZW contribution to the central charge. We shall also show how, in the Toda case, the formula (1.5) reduces to the one obtained previously [3] using BRST methods.

From the technical point of view the difficulty in quantizing the Toda-like theories is that, in the non-abelian case, the usual method of Fourier transforming to momentum-space in order to define the Fock space via normal-ordering becomes very complicated. What we shall show, however, is that a decomposition into positive and negative frequencies is sufficient to define the Fock space and to obtain our results.

2. Review of the WZW theory

Consider the WZW action [7] (for a comprehensive review see [8])

$$\kappa \int d^2x \mathcal{L}^W(g) = \kappa \left[\int d^2x \,\mathrm{tr}(g^{-1}\partial_\mu g)^2 + \frac{2}{3}\epsilon^{\alpha\beta\gamma} \int d^3x \,\mathrm{tr}(g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)(g^{-1}\partial_\gamma g) \right], \tag{2.1}$$

The chiral conserved currents are defined by

$$J = \kappa(\partial_+ g)g^{-1}, \quad \text{and} \quad \bar{J} = -\kappa g^{-1}\partial_- g, \tag{2.2}$$

where $x^\pm = \frac{1}{2}(t \pm x)$, and $\partial_\pm = \partial/\partial x^\pm = \partial_t \pm \partial_x$. These currents lie in a Lie algebra G. The *equal time* Poisson brackets generate a pair of commuting Kac-Moody algebras

$$\begin{aligned} \{J_a(x), J_b(y)\} =& f_{ab}{}^c J_c(x)\delta(x-y) + \kappa\eta_{ab}\delta'(x-y), \\ \{\bar{J}_a(x), \bar{J}_b(y)\} =& f_{ab}{}^c \bar{J}_c(x)\delta(x-y) - \kappa\eta_{ab}\delta'(x-y), \\ \{J_a(x), \bar{J}_b(y)\} =& 0, \end{aligned} \tag{2.3}$$

where $J(x) = J_a(x)\sigma^a$, and the σ^a matrices are understood to be in the *defining* representation of the Lie algebra G, and satisfy the commutation relations

$$[\sigma^a, \sigma^b] = 2f^{ab}{}_c \sigma^c. \tag{2.4}$$

η^{ab} is the Cartan metric

$$\eta^{ab} = \langle \sigma^a, \sigma^b \rangle = \mathrm{tr}\,\sigma^a\sigma^b, \qquad \eta^{ab}\eta_{bc} = \delta^a{}_c, \tag{2.5}$$

which is used to raise and lower indices in the usual way. Classically we can define Virasoro functions

$$T(x) = \frac{\langle J(x), J(x) \rangle}{2\kappa} = \frac{\eta^{ab} J_a(x) J_b(x)}{2\kappa}, \quad \bar{T}(x) = \frac{\eta^{ab} \bar{J}_a(x) \bar{J}_b(x)}{2\kappa}, \tag{2.6}$$

which satisfy commuting Virasoro algebras

$$\begin{aligned} \{T(x), T(y)\} =& 2T(x)\delta'(x-y) + T'(x)\delta(x-y), \\ \{\bar{T}(x), \bar{T}(y)\} =& -2\bar{T}(x)\delta'(x-y) - \bar{T}'(x)\delta(x-y), \\ \{T(x), \bar{T}(y)\} =& 0, \end{aligned} \tag{2.7}$$

where the prime $'$ refers to the spatial derivative ∂_x.

In the quantum theory, eqs. (2.3) are replaced by the fundamental (equal time) commutation relations

$$\begin{aligned} [J_a(x), J_b(y)] =& i\hbar f_{ab}{}^c J_c(x)\delta(x-y) + i\hbar\kappa\eta_{ab}\delta'(x-y), \\ [\bar{J}_a(x), \bar{J}_b(y)] =& i\hbar f_{ab}{}^c J_c(x)\delta(x-y) - i\hbar\kappa\eta_{ab}\delta'(x-y), \\ [J_a(x), \bar{J}_b(y)] =& 0. \end{aligned} \tag{2.8}$$

At the quantum level we must use normal ordered operators (or else perform some other regularization). Now consider the operator (which is proportional to the Virasoro operator $T(x)$ for a pure WZW theory)

$$t(x) = \eta^{ab} : J_a(x) J_b(x) : . \tag{2.9}$$

The basic commutators of normal ordered products are derived in appendix A; here we just quote the results. Now

$$[t(x), J_p(y)] = i\hbar \left[2\kappa J_p(x) + \frac{\hbar}{2\pi} \eta^{ab} f_{ap}{}^c f_{cb}{}^d J_d(x) \right] \delta'(x-y). \tag{2.10}$$

If the group is *simple* or *abelian* eqn. (2.10) simplifies to

$$[t(x), J_p(y)] = i\hbar(2\kappa + \hbar\mathcal{G}) J_p(x)\delta'(x-y), \tag{2.11}$$

where $\mathcal{G}$ defined by

$$\eta^{ab} f_{ap}{}^c f_{cb}{}^d = 2\pi\mathcal{G}\delta_p{}^d, \tag{2.12}$$

or $\big[\sigma^a, [\sigma_a, p]\big] = -8\pi\mathcal{G}p$ for $p \in G$, ie. $-8\pi\mathcal{G}$ is the quadratic Casimir in the adjoint representation. In particular, for $sl(n)$, $-8\pi\mathcal{G} = \frac{1}{2}n$. Define the Virasoro operator

$$T(x) = \frac{t(x)}{2\kappa + \hbar\mathcal{G}}, \tag{2.13}$$

then

$$[T(x), J_p(y)] = i\hbar J_p(x)\delta'(x-y). \tag{2.14}$$

This is just the statement that the KM current has conformal spin one. It is straightforward to show that $T(x)$ satisfies the following Virasoro algebra (again see appendix A)

$$[T(x), T(y)] = 2i\hbar T(x)\delta'(x-y) + i\hbar T'(x)\delta(x-y) - \frac{i\hbar C}{24\pi}\delta'''(x-y), \tag{2.15}$$

with the central charge

$$C = \frac{2\hbar\kappa \dim G}{2\kappa + \hbar\mathcal{G}}. \tag{2.16}$$

$T(x)$ as defined by eqn. (2.13) is not the most general Virasoro operator, consider

$$L(x) = T(x) + I(x), \tag{2.17}$$

where the conformal "improvement" term $I(x)$ is defined by

$$I(x) = \langle \mathcal{I}, J'(x) \rangle, \tag{2.18}$$

and $\mathcal{I} \in G$. Now $L(x)$ still satisfies the Virasoro algebra (eqn. (2.15)), but with a modified central charge

$$C = \frac{2\hbar\kappa \dim\ G}{2\kappa + \hbar\mathcal{G}} + 24\pi\kappa\langle \mathcal{I}, \mathcal{I} \rangle. \tag{2.19}$$

3. Conformal character of g(x)

When considering interactions, we will need the Poisson bracket (and quantum commutator) relations regarding $g(x)$. Classically $g(x)$ satisfies

$$2\kappa g'(x) = (\partial_+ - \partial_-)g = J(x)g(x) + g(x)\bar{J}(x). \tag{3.1}$$

Together with appropriate boundary conditions, we can regard eqn. (3.1) as defining $g(x)$ in terms of the KM variables. $g(x)$ obeys the following Poisson bracket relations

$$\{J_p(x), g(y)\} = -\tfrac{1}{2}\sigma_p g(y)\delta(x-y), \qquad \{\bar{J}_p(x), g(y)\} = \tfrac{1}{2}g(y)\sigma_p\delta(x-y), \tag{3.2}$$

and

$$\{[g(x)]_{ab}, [g(y)]_{cd}\} = 0. \tag{3.3}$$

Equation (3.3) is sometimes written as

$$\{g(x) \overset{\otimes}{,} g(y)\} = 0. \tag{3.4}$$

Note that the $\{g \overset{\otimes}{,} g\}$ bracket is non-trivial on the light cone, and involves the Yang-Baxter r-matrices [9].

We cannot transplant eqn. (3.1) directly into the quantum theory since it is not a normal ordered equation. The quantum analogue of eqn. (3.1) is [10]

$$\beta g'(x) = \sigma^a : J_a(x)g(x) : + : \bar{J}_a(x)g(x) : \sigma^a, \tag{3.5}$$

where

$$\beta = 2\kappa + \hbar\mathcal{G}. \tag{3.6}$$

We show in appendix B that eqn. (3.5) together with the KM algebra eqn. (2.8) is compatible with the basic commutators

$$[J_p(x), g(y)] = -\tfrac{i}{2}\hbar\sigma_p g(y)\delta(x-y), \qquad [\bar{J}_p(x), g(y)] = \tfrac{i}{2}\hbar g(y)\sigma_p\delta(x-y), \tag{3.7}$$

and

$$[g(x) \overset{\otimes}{,} g(y)] = 0. \tag{3.8}$$

If one attempted to define a $g(x)$ operator with a value of β other than that given by eqn. (3.6) the $[J_p(x), g(y)]$ commutator would be non-local in the non-abelian case.

We can also define a normal ordered $g^{-1}(x)$ operator via

$$\beta\partial_x g^{-1}(x) = -:J_a(x)g^{-1}(x):\sigma^a - \sigma^a:\bar{J}_a(x)g^{-1}(x):, \tag{3.9}$$

and this satisfies

$$[J_p(x), g^{-1}(y)] = \tfrac{i}{2}\hbar g^{-1}(y)\sigma_p\delta(x-y), \qquad [\bar{J}_p(x), g^{-1}(y)] = -\tfrac{i}{2}\hbar\sigma_p g^{-1}(y)\delta(x-y). \tag{3.10}$$

Note that $g^{-1}(x)$ is *not* the inverse of $g(x)$ (except in the limit $\hbar \to 0$), since the inverse of $g(x)$ would be anti-normal ordered.

Classically we have

$$\{T(x), g(y)\} + \{g(x), T(y)\} = 0, \tag{3.11}$$

which is just the statement that $g(x)$ is a conformal scalar. In the quantum theory $g(x)$ acquires an anomalous dimension (see Appendix B); the analogue of eqn. (3.11) is

$$[T(x), g(y)] + [g(x), T(y)] = -\frac{i\hbar^2 c}{8\pi(2\kappa+\hbar\mathcal{G})}\Big[2g(x)\delta'(x-y) + g'(x)\delta(x-y)\Big], \tag{3.12}$$

where c is the quadratic Casimir in the defining representation, ie.

$$cI = \eta^{ab}\sigma_a\sigma_b, \tag{3.13}$$

I being the identity matrix, and $T(x)$ is the correctly normalized kinetic term given by eqn. (2.13). We also have

$$[T(x), g^{-1}(y)] + [g^{-1}(x), T(y)] = -\frac{i\hbar^2 c}{8\pi(2\kappa+\hbar\mathcal{G})}\Big[2g^{-1}(x)\delta'(x-y) + \partial_x g^{-1}(x)\delta(x-y)\Big]. \tag{3.14}$$

Note that the sign on the RHS of eqn. (3.14) is the *same* sign as on the RHS of eqn. (3.12), whereas if $g^{-1}(x)$ was the inverse of $g(x)$ (rather than just a normal ordered version of the inverse) the signs would be opposite.

4. Classical Interaction Potentials

We now consider *interacting* WZW models which are conformally invariant. Let us add a "potential" term to the ordinary WZW Lagrangian eqn. (2.1).

$$\mathcal{L} = \mathcal{L}^W(g) - 2V(g). \tag{4.1}$$

Classically, this will entail adding a potential term to the canonical energy momentum tensor

$$T^{\mu\nu}_{can}(x) = T^{\mu\nu}_W(x) + 2g^{\mu\nu}V(x), \tag{4.2}$$

where $T^{\mu\nu}_W$ refers to the free WZW canonical energy momentum tensor. One can see immediately that *any* potential term will spoil the tracelessness of $T^{\mu\nu}_{can}(x)$. Moreover, it is not possible to construct a Virasoro algebra based on the canonical EM tensor, ie. $L_{can}(x) = \frac{1}{2}[T^{00}_{can}(x) + T^{01}_{can}(x)]$ and $\bar{L}_{can}(x) = \frac{1}{2}[T^{00}_{can}(x) - T^{01}_{can}(x)]$ do not satisfy the Virasoro algebra. To see this, consider

$$L_{can}(x) = T(x) + V(x), \qquad \bar{L}_{can}(x) = \bar{T}(x) + V(x), \tag{4.3}$$

where $T(x)$ and $\bar{T}(x)$ refer to the canonical pure WZW contribution. We assume that

$$\{V(x), V(y)\} = 0, \tag{4.4}$$

ie. we suppose that our potential $V(x)$ contains no derivatives of $g(x)$, and then eqn. (4.4) follows from eqn. (3.4). Now since

$$\{T(x), V(y)\} + \{V(x), T(y)\} = 0, \tag{4.5}$$

which is equivalent to the statement that $V(x)$ has conformal spin zero, then

$$\begin{aligned}\{L(x), L(y)\} =& 2T(x)\delta'(x-y) + T'(x)\delta(x-y)\\ \neq& 2L(x)\delta'(x-y) + L'(x)\delta(x-y) - \frac{C}{24\pi}\delta'''(x-y).\end{aligned} \tag{4.6}$$

However, we know from section 2 that the canonical EM tensor is not the most general EM tensor for the *pure* WZW theory which generates a Virasoro algebra. In the pure WZW case we are free to add an improvement term $I(x)$ whose only effect is to modify the central charge C. As alternatives to L_{can} and $\bar{L}_{can}$ consider the following Virasoro functions (such a procedure was used by Curtright and Thorn to improve the Liouville energy-momentum tensor at the classical and quantum level [11], see also [12])

$$L(x) = T(x) + V(x) + I(x), \qquad \bar{L}(x) = \bar{T}(x) + V(x) + \bar{I}(x), \tag{4.7}$$

where as in section 2, $I(x)$ and $\bar{I}(x)$ are understood to be linear in the spatial derivatives of the KM currents, with the Poisson bracket relations

$$\{I(x), I(y)\} = -\frac{C}{24\pi}\delta'''(x-y), \qquad \{\bar{I}(x), \bar{I}(y)\} = \frac{C}{24\pi}\delta'''(x-y),$$

$$\{T(x), I(y)\} = I(x)\delta'(x-y), \qquad \{\bar{T}(x), \bar{I}(y)\} = -\bar{I}(x)\delta'(x-y). \tag{4.8}$$

Now

$$\begin{aligned}\{L(x), L(y)\} =& 2\big(T(x) + I(x)\big)\delta'(x-y) + \big(T'(x) + I'(x)\big)\delta(x-y) \\ &+ \{L(x), V(y)\} + \{V(x), L(y)\} + \{I(x), I(y)\} \\ =& 2L(x)\delta'(x-y) + L'(x)\delta(x-y) - \frac{C}{24}\delta'''(x-y),\end{aligned} \tag{4.9}$$

if

$$\{L(x), V(y)\} + \{V(x), L(y)\} = 2V(x)\delta'(x-y) + V'(x)\delta(x-y). \tag{4.10}$$

Using eqn. (4.5) this simplifies to

$$\{I(x), V(y)\} + \{V(x), I(y)\} = 2V(x)\delta'(x-y) + V'(x)\delta(x-y). \tag{4.11}$$

or

$$\{I(x), V(y)\} = V(y)\delta'(x-y). \tag{4.12}$$

That is if we can find an improvement term that satisfies the above equation (there will be a similar condition concerning $\bar{I}(x)$) then we have a conformally invariant interaction. The classical central extension of the Virasoro algebra is entirely due to the $\{I(x), I(y)\}$ bracket.

5. Quantum Interaction Potentials

We extend the considerations of the previous section to the quantum level. As in the classical case, the task of establishing conformal invariance and computing the central charge amounts to the problem of finding suitable improvement operators $I(x)$ and $\bar{I}(x)$. However there is a slight complication, in that $[T(x), V(y)] + [V(x), T(y)]$ is in general non-zero (this follows from eqn. (3.12)). From eqn. (3.8) (and the assumption that our interaction potential contains no derivatives) we have

$$[V(x), V(y)] = 0, \tag{5.1}$$

and so the conformal invariance condition analogous to eqn. (4.10) is

$$[L(x), V(y)] + [V(x), L(y)] = 2i\hbar V(x)\delta'(x-y) + i\hbar V'(x)\delta(x-y). \tag{5.2}$$

Let us assume, however, that the potential can be "diagonalized" with respect to the action of $T(x)$, ie.

$$V(y) = \sum_a V_a(y), \tag{5.3}$$

where each of the components V_a have a definite conformal weight $\hbar\lambda_a$,

$$[T(x), V_a(y)] + [V_a(x), T(y)] = i\hbar^2\lambda_a \left[2V_a(x)\delta'(x-y) + V_a'(x)\delta(x-y)\right]. \tag{5.4}$$

There is no summation over a in the last equation. We will demonstrate such a decomposition for Toda-like theories in the next section. Substituting eqn. (5.4) into eqn. (5.2), the conformal invariance condition reduces to

$$[I(x), V_a(y)] + [V_a(x), I(y)] = i\hbar(1-\hbar\lambda_a)\left[2V_a(x)\delta'(x-y) + V_a'(x)\delta(x-y)\right], \qquad (5.5)$$

or

$$[I(x), V_a(y)] = i\hbar(1-\hbar\lambda_a)V_a(y)\delta'(x-y), \qquad (5.6)$$

which is the quantum analogue of eqn. (4.12). Thus, the problem reduces to finding an improvement term $I(x)$ which satisfies eqn. (5.6). In the next section we will show that for "Toda-like" theories this problem is easily solved.

6. Toda-like Theories

We now consider the Toda-like Lagrangians defined by eqn. (1.3). As mentioned in the introduction, the G_o algebra splits into a central piece G_c and a non-abelian piece $\sum_\alpha G_\alpha$ which is just the sum of the $sl(n_\alpha)$ components. It is convenient to use a (non-orthogonal) basis for G_c with the properties

$$[\sigma_a, M_\pm] = \pm M_\pm^{(a)}, \quad \text{and} \quad M_o = \sum_a \sigma_a. \qquad (6.1)$$

The explicit form of the σ_a matrices and their duals σ^a is given in appendix C. Note that the σ_a matrices also satisfy

$$[\sigma_a, N_\pm] = \pm N_\pm^{(a)}. \qquad (6.2)$$

We can write, $g_c(x)$, the central piece of $g_o(x)$ as follows

$$g_c(x) = e^{\phi^a(x)\sigma_a}. \qquad (6.3)$$

The potential term in eqn. (1.3) can be decomposed as follows

$$V\big(g_o(x)\big) = \mathrm{tr}\left(g_o N_+ g_o^{-1} N_-\right) = \sum_a V_a(g_o), \qquad (6.4)$$

where

$$V_a(g_o) = \mathrm{tr}\left(g_\alpha N_+^{(a)} g_{\alpha+1}^{-1} N_-^{(a)}\right) e^{\phi^a}, \quad (\alpha = a), \qquad (6.5)$$

where $\alpha = a$, and the g_α are the $sl(n_\alpha)$ WZW variables. The central fields $\phi^a(x)$ satisfy the Poisson bracket relation

$$\{j_b(x), \phi^a(y)\} = -\tfrac{1}{2}\delta_b{}^a\delta(x-y), \qquad (6.6)$$

which follows from eqn. (3.2) applied to G_c. The central KM currents satisfy

$$\{j_a(x), j_b(y)\} = \kappa_{ab}\delta'(x-y). \qquad (6.7)$$

When normal ordering the V_a potentials, one can normal order the g_α, $g_{\alpha+1}^{-1}$ and e^{ϕ^a} separately using the methods described in appendix B. The free quantum Virasoro operator takes the usual form

$$T(x) = \sum_\alpha \frac{1}{2\kappa_\alpha + \hbar\mathcal{G}_\alpha} \mathrm{tr}\left[: J\big(g_\alpha(x)\big)^2 :\right] + \frac{1}{2}\sum_{ab} \kappa^{ab} : j_a(x) j_b(x) : \tag{6.8}$$

where κ^{ab} is the inverse of κ_{ab}. The decomposition $V = \sum_a V_a$ given in eqn. (6.4) is already of the "diagonal" form, in that the V_a satisfy eqn. (5.4), with

$$8\pi\lambda_a = -\frac{1}{2}\kappa^{aa} - \frac{c_\alpha}{2\kappa_\alpha + \hbar\mathcal{G}_\alpha} - \frac{c_{\alpha+1}}{2\kappa_{\alpha+1} + \hbar\mathcal{G}_{\alpha+1}}, \quad (\alpha = a). \tag{6.9}$$

Here, the $-c_\alpha/(2\kappa + \hbar\mathcal{G}_\alpha)$ term is due to the anomalous conformal weight of $g_\alpha(x)$ (see eqn. (3.12)), the $-c_{\alpha+1}/(2\kappa + \hbar\mathcal{G}_{\alpha+1})$ is the contribution from $g_{\alpha+1}^{-1}$ (eqn. (3.14)), and the $-\frac{1}{2}\kappa^{aa}$ term is the due to the (abelian) $: e^{\phi^\alpha} :$ term in the normal ordered version of eqn. (6.5). For $sl(n_\alpha)$ we have $\mathcal{G}_\alpha = -n_\alpha/(16\pi)$, and † $c_\alpha = (n_\alpha^2 - 1)/n_\alpha$. For principal embeddings (abelian Toda-like theories) we have $c_\alpha = c_{a+1} = 0$ and thus λ_a reduces to $-\kappa^{aa}/(16\pi)$. Note that even in this case the λ_a are not, in general, uniform in a and thus the components V_a of the potential have different anomalous conformal weights.

The system is conformally invariant if there exists an $I(x)$ which satisfies eqn. (5.6). It is easy to see that $I(x) = \langle \mathcal{I}, J'(x) \rangle$, with

$$\mathcal{I} = \sum_a (1 - \hbar\lambda_a)\sigma_a = M_o - \hbar M_q, \tag{6.10}$$

is a solution to eqn. (5.6). The centre of the Virasoro algebra is evidently

$$C = \hbar\left[\dim G_c + \sum_\alpha \frac{2\kappa_\alpha \dim G_\alpha}{2\kappa_\alpha + \hbar\mathcal{G}_\alpha}\right] + 24\pi \sum_{ab} \kappa_{ab} \mathrm{tr}\big(\sigma^a I\big)\mathrm{tr}\big(\sigma^b I\big) \tag{6.11}$$

In the abelian case this reduces to

$$C = \hbar \dim H + 24\pi \sum_{ab} \kappa_{ab} \mathrm{tr}\big(\sigma^a \mathcal{I}\big)\mathrm{tr}\big(\sigma^b \mathcal{I}\big), \tag{6.12}$$

H being the Cartan subalgebra of G.

7. Comparison with the Reduction Formula

Classically, the Toda case is obtained from the general one by imposing the uniformity relations $\kappa_{ab} = \kappa\,\mathrm{tr}(\sigma_a\sigma_b)$ and $\kappa_\alpha = \kappa$. At the quantum level, however, the uniformity conditions are [3,5]

$$2\kappa_{ab} = (2\kappa + \hbar\mathcal{G})\mathrm{tr}(\sigma_a\sigma_b) \quad \text{and} \quad 2\kappa_\alpha + \hbar\mathcal{G}_\alpha = 2\kappa + \hbar\mathcal{G}, \tag{7.1}$$

† Let σ_a be a basis for $sl(n_\alpha)$ in the defining representation. Taking the trace of $c_\alpha I = \sigma^a\sigma_a$ gives $n_\alpha c_\alpha = \dim G_\alpha = n_\alpha^2 - 1$.

where $-8\pi\mathcal{G}$ is the quadratic Casimir for $sl(N)$ in the adjoint representation, ie. $\mathcal{G} = -N/(16\pi)$.

On using the conditions (7.1) the expression (6.10) for the central charge simplifies to

$$C = \hbar\left[\dim G_c + \sum_\alpha \frac{2\kappa_\alpha \dim G_\alpha}{2\kappa_\alpha + \hbar\mathcal{G}_\alpha}\right] + 12\pi(2\kappa + \hbar\mathcal{G})\mathrm{tr}\Big[\big(M_o - \hbar M_q\big)^2\Big], \qquad (7.2)$$

where $M_o - \hbar M_q = \sum_a (1 - \hbar\lambda_a)\,\sigma_a$. Using the expression for σ^a given in appendix C we have

$$\kappa^{aa} = \frac{2}{(2\kappa + \hbar\mathcal{G})}\mathrm{tr}(\sigma^a\sigma^a) = \frac{2}{(2\kappa + \hbar\mathcal{G})}\left(\frac{1}{n_\alpha} + \frac{1}{n_{\alpha+1}}\right), \quad (\alpha = a), \qquad (7.3)$$

and $c_\alpha = (n_\alpha^2 - 1)/n_\alpha$. Therefore eqn. (6.9) for λ_a simplifies to

$$8\pi\lambda_a = -\frac{1}{(2\kappa + \hbar\mathcal{G})}\big(n_\alpha + n_{\alpha+1}\big), \quad (\alpha = a). \qquad (7.4)$$

The formula obtained previously using BRST methods is [3]

$$C = \hbar\dim G_o + 12\pi(2\kappa + \hbar\mathcal{G})\mathrm{tr}\left[M_o + \frac{\hbar m_o}{4\pi(2\kappa + \hbar\mathcal{G})}\right]^2, \qquad (7.5)$$

where M_o and m_o are the grading operators corresponding to the actual and the principle $sl(2)$ embeddings respectively. The formulae (7.2) and (7.5) are rather similar but they cannot be compared immediately because (7.5) is not in the form in which the part coming from the free theory is separated from the part due to the potential. However, (7.5) can be converted to the canonical form by decomposing m_o into its block components. Let $m_o^{(\alpha)}$ be the grading operators for the principal $sl(2)$-embeddings within blocks and $\bar{m}_\alpha$ denote the block-average of m_o within the $sl(n_\alpha)$ block. Then, since the spectra of both m_o and $m_o^{(a)}$ are integer spaced and the block-trace of $m_o^{(a)}$ is zero, we have

$$m_o = \sum_\alpha m_o^{(a)} + \bar{M} \quad \text{where} \quad \bar{M} = \bar{m}_\alpha I_\alpha. \qquad (7.6)$$

The tracelessness of the $m_o^{(a)}$ within blocks ensures that the $m_o^{(a)}$ are trace-orthogonal to M_o and to $\bar{M}$ and, using this trace-orthogonality, we can write eqn. (7.5) in the form

$$C = \hbar\dim G_c + \hbar\sum_\alpha\left[\dim G_\alpha + \frac{3\hbar\mathrm{tr}\big(m_o^{(\alpha)}\big)^2}{4\pi(2\kappa_\alpha + \hbar\mathcal{G}_\alpha)}\right] + 12\pi(2\kappa + \hbar\mathcal{G})\mathrm{tr}\left[M_o + \frac{\hbar\bar{M}}{4\pi(2\kappa + \hbar\mathcal{G})}\right]^2. \qquad (7.7)$$

Using $m_o^{(\alpha)} = \mathrm{diag}\left[\frac{1}{2}(n_\alpha - 1), \frac{1}{2}(n_\alpha - 3), ..., -\frac{1}{2}(n_\alpha - 1)\right]$ it is straightforward to show that $\mathrm{tr}\big(m_o^{(\alpha)}\big)^2 = \frac{1}{12}n_\alpha(n_\alpha + 1)(n_\alpha - 1)$. This can be rewritten as $\mathrm{tr}\big(m_o^{(\alpha)}\big)^2 = \frac{1}{12}n_\alpha\dim G_\alpha = -\frac{4}{3}\pi\mathcal{G}_\alpha\dim G_\alpha$. Inserting the last equality into eqn. (7.7) gives

$$C = \hbar\left[\dim G_c + \sum_\alpha \frac{2\kappa_\alpha \dim G_\alpha}{2\kappa_\alpha + \hbar\mathcal{G}_\alpha}\right] + 12\pi(2\kappa + \hbar\mathcal{G})\mathrm{tr}\left[M_o + \frac{\hbar\bar{M}}{4\pi(2\kappa + \hbar\mathcal{G})}\right]^2. \qquad (7.8)$$

In (7.8) the part of the centre coming from the free WZW theory is separated (in the square brackets) as required.

We can now compare (7.8) and (7.2) and we see that they are the same provided that $2\bar{M} = -(2\kappa + \hbar\mathcal{G})M_q$. But from (7.6) and (7.4) we see that this holds if

$$2\sum_\alpha \bar{m}_\alpha I_\alpha = -\sum_a (n_\alpha + n_{\alpha+1})\sigma_a. \tag{7.9}$$

But from the definition of σ_a given in appendix C we have

$$\begin{aligned}\sum_\alpha \bar{m}_\alpha I_\alpha &= \sum_\alpha \left[\bar{m}_\alpha(\sigma_a - \sigma_{a-1}) - \frac{\bar{m}_\alpha n_\alpha}{N} I\right] \\ &= \sum_\alpha \bar{m}_\alpha(\sigma_a - \sigma_{a-1}) = \sum_\alpha (\bar{m}_\alpha - \bar{m}_{\alpha+1})\sigma_a = -\frac{1}{2}\sum_\alpha (n_\alpha + n_{\alpha+1})\sigma_a,\end{aligned} \tag{7.10}$$

where we have used the fact that $\sum \bar{m}_\alpha n_\alpha = \mathrm{tr}(\bar{M}) = 0$. Thus the general formula for the centre reduces to the BRST formula in the Toda case, as required. We also see that the formula (7.8) and not (7.5) is the more natural one for the Toda theory.

8. Conclusions

In this paper we have considered the canonical quantization of conformally invariant interacting WZW theories. In particular, we have considered "Toda-like" models, which for special values of the couplings reduce to the well known Toda (abelian and non-abelian) theories. In the canonical approach, the computation of the Virasoro central charge amounts to the problem of finding a suitable "improved" energy-momentum tensor. In the Toda case, our formula for the central charge reduces to the expression previously obtained in the BRST approach.

Acknowledgements

We thank W.McGlinn, I.Sachs and C.Wiesendanger for useful discussions. L.O'R. is grateful to the University of Notre Dame for their kind hospitality and financial support while much of this work was being performed.

Appendix A: Normal Ordering

In order to define a Virasoro operator in the quantum case, we introduce a normal ordering prescription. This is usually accomplished by identifying the Fourier modes of $J(x)$ and $\bar{J}(x)$ as either creation or annihilation operators with respect to a vacuum state $|0\rangle$. Then an operator is normal ordered if all annihilation operators are placed to the right of creation operators. However, we can perform the splitting of $J(x)$ into creation and annihilation operators while remaining in *position* space as follows. Write

$$J(x) = J^+(x) + J^-(x), \quad \text{and} \quad \delta(x) = \delta^+(x) + \delta^-(x), \tag{A.1}$$

where

$$J^{\pm}(x) = \int_{-\infty}^{\infty} dy\, \delta^{\pm}(x-y) J(y), \tag{A.2}$$

and

$$\delta^{\pm}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \theta(\mp k) e^{ik(x\mp i\epsilon)} = \pm \frac{i}{2\pi} \frac{1}{(x \mp i\epsilon)}, \tag{A.3}$$

where $\epsilon > 0$. Trivially we also have $\delta^{\pm}(-x) = \delta^{\mp}(x)$ and $\delta^{\pm\prime}(-x) = -\delta^{\mp\prime}(x)$, where $\delta^{\pm\prime}(x) = \partial_x \delta^{\pm}(x)$. We now interpret $J^-(x)$ and $J^+(x)$ as creation and annihilation operators, respectively

$$J^+(x)|0\rangle = 0, \qquad \langle 0|J^-(x) = 0, \tag{A.4}$$

and as usual an operator is normal ordered if all J^+ operators are placed to the right of J^- operators. Similarly we decompose $\bar{J}(x) = \bar{J}^-(x) + \bar{J}^+(x)$. However the normal ordering of the $\bar{J}$ operators is done in the *opposite* way to the J operators; an operator is normal ordered if all $\bar{J}^-$ operators are placed to the right of $\bar{J}^+$ operators. We will make extensive use of the following identities, which follow immediately from eqn. (A.3)

$$\left[\delta^+(x)\right]^2 - \left[\delta^-(x)\right]^2 = \frac{i}{2\pi}\delta'(x) \tag{A.5}$$

$$\left[\delta^{+\prime}(x)\right]^2 - \left[\delta^{-\prime}(x)\right]^2 = \frac{i}{12\pi}\delta'''(x). \tag{A.6}$$

Multiplying the KM algebra eqs. (2.8) by $\delta^{\pm}(z-x)$ and integrating with respect to x gives the very useful relation

$$[J_a^{\pm}(z), J_b(y)] = i\hbar f_{ab}{}^{c} J_c(y)\delta^{\pm}(z-y) + i\hbar\kappa\eta_{ab}\delta^{\pm\prime}(z-y), \tag{A.7}$$

and similarly for $[\bar{J}_a^{\pm}(z), \bar{J}_b(y)]$. Notice the current on the RHS of eqn. (A.7) is the full current, and not just the creative or destructive piece.

Writing the $t(x)$ operator defined by eqn. (2.9) in terms of $J^+(x)$ and $J^-(x)$

$$t(x) = \eta^{ab} : J_a(x)J_b(x) := \eta^{ab}\left[J_a^+(x)J_b^+(x) + J_a^-(x)J_b^-(x) + 2J_a^-(x)J_b^+(x)\right]. \tag{A.8}$$

We derive eqn. (2.10) for the $[t(x), J_p(y)]$ commutator

$$\begin{aligned}[t(x), J_p(y)] =& 2i\hbar\kappa J_p(x)\delta'(x-y) \\ &+ i\hbar\eta^{ab} f_{ap}{}^{c}\Big\{\left[J_b^+(x)J_c(y) + J_c(y)J_b^+(x) + 2J_b^-(x)J_c(y)\right]\delta^+(x-y) \\ &+ \left[J_b^-(x)J_c(y) + J_c(y)J_b^-(x) + 2J_c(y)J_b^+(x)\right]\delta^-(x-y)\Big\}.\end{aligned} \tag{A.9}$$

Some of the terms on the right hand side of eqn. (A.9) are not correctly normal ordered. We rewrite them in terms of normal ordered objects,

$$\begin{aligned}J_b^+(x)J_c(y) + J_c(y)J_b^+(x) + 2J_b^-(x)J_c(y) &= :J_b(x)J_c(y) + J_c(y)J_b(x): + [J_b(x), J_c^-(y)], \\ J_b^-(x)J_c(y) + J_c(y)J_b^-(x) + 2J_c(y)J_b^+(x) &= :J_b(x)J_c(y) + J_c(y)J_b(x): + [J_c^+(y), J_b(x)],\end{aligned} \tag{A.10}$$

and here

$$: J_b(x)J_c(y) := J_b^+(x)J_c^+(y) + J_b^-(x)J_c^-(y) + J_b^-(x)J_c^+(y) + J_c^-(y)J_b^+(x). \qquad (A.11)$$

Thus

$$\begin{aligned}
[t(x), J_p(y)] =& 2i\hbar\kappa J_p(x)\delta'(x-y) \\
&+ i\hbar\eta^{ab} f_{ap}{}^{c}\Big\{\big(: J_b(x)J_c(y) + J_c(y)J_b(x) : + [J_b(x), J_c^-(y)]\big)\delta^+(x-y) \\
&\quad + \big(: J_b(x)J_c(y) + J_c(y)J_b(x) : + [J_c^+(y), J_b(x)]\big)\delta^-(x-y)\Big\} \\
=& 2i\hbar\kappa J_p(x)\delta'(x-y) + i\hbar\eta^{ab} f_{ap}{}^{c} \left(: J_b(x)J_c(y) + J_c(y)J_b(x) :\right)\delta(x-y) \\
&+ \hbar^2\eta^{ab} f_{ap}{}^{c} f_{cb}{}^{d} J_d(x)\big\{[\delta^+(x-y)]^2 - [\delta^-(x-y)]^2\big\} \\
=& i\hbar\left[2\kappa J_p(x) + \frac{\hbar}{2\pi}\eta^{ab} f_{ap}{}^{c} f_{cb}{}^{d} J_d(x)\right]\delta'(x-y),
\end{aligned} \qquad (A.12)$$

which is eqn. (2.10). In equation (A.12) we have used the identity (A.5). If the group is *simple* or *abelian* then eqn. (A.12) simplifies to

$$[t(x), J_p(y)] = i\hbar(2\kappa + \hbar\mathcal{G})J_p(x)\delta'(x-y), \qquad (A.13)$$

where $\mathcal{G}$ is defined by eqn. (2.12).

We now compute $[t(x), t(y)]$. Using the $[t(x), J(y)]$ result it is easy to see that

$$[t(x), J_p^{\pm}(y)] = i\hbar(2\kappa + \hbar\mathcal{G})J_p(x)\delta^{\mp\prime}(x-y). \qquad (A.14)$$

Now

$$\begin{aligned}
[t(x), t(y)] =& \eta^{ab}\Big\{J_a^+(y)[t(x), J_b^+(y)] + [t(x), J_a^-(y)]J_b^-(y) + [t(x), J_a^+(y)]J_b^+(y) \\
&+ J_a^-(y)[t(x), J_b^-(y)] + 2J_a^-(y)[t(x), J_b^+(y)] + 2[t(x), J_a^-(y)]J_b^+(y)\Big\}.
\end{aligned} \qquad (A.15)$$

The first two entries on the RHS of the last equality are not normal ordered, and so they must be "re-normal ordered" as in eqn. (A.10). Which gives

$$\begin{aligned}
[t(x), t(y)] =& \eta^{ab}\Big\{2[t(x), J_a^+(y)]J_b^+(y) + 2J_a^-(y)[t(x), J_b^-(y)] + 2J_a^-(y)[t(x), J_b^+(y)] \\
&+ 2[t(x), J_a^-(y)]J_b^+(y) + \big[J_a^+(y), [t(x), J_b^+(y)]\big] - \big[J_a^-(y), [t(x), J_b^-(y)]\big]\Big\} \\
=& i\hbar(2\kappa + \hbar\mathcal{G})\eta^{ab}\left(: J_a(x)J_b(y) + J_a(y)J_b(x) :\right)\delta'(x-y) \\
&- \hbar^2\kappa\,(2\kappa + \hbar\mathcal{G})\dim G\left\{[\delta^{+\prime}(x-y)]^2 - [\delta^{-\prime}(x-y)]^2\right\} \\
=& 2i\hbar(2\kappa + \hbar\mathcal{G})t(x)\delta'(x-y) + i\hbar(2\kappa + \hbar\mathcal{G})t'(x)\delta(x-y) \\
&- \frac{i\hbar^2\kappa}{12\pi}(2\kappa + \hbar\mathcal{G})\dim G\,\delta'''(x-y),
\end{aligned} \qquad (A.16)$$

using the identity eqn. (A.6). Using eqn. (2.13), the Virasoro algebra eqn. (2.15) follows immediately. We note that the central extension in eqn. (2.15) is just the contribution from the double commutator terms that arise from "re-normal" ordering, ie.

$$i\hbar C\delta'''(x-y) = 24\pi\eta^{ab}\Big\{\big[J_a^+(y),[J_b^+(y),T(x)]\big] - \big[J_a^-(y),[J_b^-(y),T(x)]\big]\Big\}. \qquad (A.17)$$

Appendix B: Properties of g(x)

We now show that $g(x)$ satisfying the normal ordered equation (3.5) is compatible with eqs. (3.7) only when $\beta = 2\kappa + \hbar\mathcal{G}$. In terms of $J^{\pm}(x)$, eqn. (3.5) reads

$$\beta g'(x) = \left[\sigma^a J_a^-(x)g(x) + \sigma^a g(x)J_a^+(x) + \bar{J}_a^+(x)g(x)\sigma^a + g(x)\bar{J}^-(x)\sigma^a\right]. \qquad (B.1)$$

Now

$$\begin{aligned}\beta[J_p(z),g'(x)] =&\sigma^a[J_p(z),J_a^-(x)]g(x) + \sigma^a J_a^-(x)[J_p(z),g(x)] + \sigma^a[J_p(z),g(x)]J_a^+(x)\\ &+ \sigma^a g(x)[J_p(z),J_a^+(x)] + \bar{J}_a^+(x)[J_p(z),g(x)]\sigma^a + [J_p(z),g(x)]\bar{J}_a^-(x)\sigma^a\\ =&i\hbar\kappa\sigma_p g(x)\delta'(z-x)\\ &+ i\hbar f_{pa}{}^d\sigma^a J_d(z)g(x)\delta^+(z-x) + i\hbar f_{pa}{}^d\sigma^a g(x)J_d(z)\delta^-(z-x)\\ &+ \sigma^a J_a^-(x)[J_p(z),g(x)] + \sigma^a[J_p(z),g(x)]J_a^+(x)\\ &+ \bar{J}_a^+(x)[J_p(z),g(x)]\sigma^a + [J_p(z),g(x)]\bar{J}_a^-(x)\sigma^a.\end{aligned} \qquad (B.2)$$

Re-normal ordering the right hand side of eqn. (B.2), we have

$$\begin{aligned}\beta\partial_x[J_p(z),g(x)] =&i\hbar\kappa\sigma_p\delta'(z-x)g(x) + i\hbar f_{pa}{}^d\sigma^a\left[J_d^-(z)g(x) + g(x)J_d^+(z)\right]\delta(x-z)\\ &+ i\hbar\sigma^a f_{pa}{}^d\left\{[J_d^+(z),g(x)]\delta^+(z-x) + [g(x),J_d^-(z)]\delta^-(z-x)\right\}\\ &+ \sigma^a J_a^-(x)[J_p(z),g(x)] + \sigma^a[J_p(z),g(x)]J_a^+(x)\\ &+ \bar{J}_a^+(x)[J_p(z),g(x)]\sigma^a + [J_p(z),g(x)]\bar{J}_a^-(x)\sigma^a.\end{aligned} \qquad (B.3)$$

One can derive a similar equation for $\partial_x[\bar{J}_p(z),g(x)]$. We can regard eqn. (B.3) as a differential equation for $[J_p(z),g(x)]$. Now we will show that eqs. (3.7), ie.

$$[J_p(z),g(x)] = -\tfrac{i}{2}\hbar\sigma_p g(x)\delta(z-x), \qquad [\bar{J}_p(z),g(x)] = \tfrac{i}{2}\hbar g(x)\sigma_p\delta(z-x), \qquad (B.4)$$

satisfy eqn. (B.4) for a *unique* value of β. Inserting eqs. (B.4) into eqn. (B.3) gives

$$\begin{aligned}-\tfrac{i}{2}\hbar\beta\sigma_p\partial_x\left[g(x)\delta(z-x)\right] =&\tfrac{i}{2}\hbar(2\kappa+\hbar\mathcal{G})\sigma_p g(x)\delta'(z-x)\\ &+ i\hbar\sigma^a f_{pa}{}^d\left[J_d^-(z)g(x) + g(x)J_d^+(z)\right]\delta(x-z)\\ &- \tfrac{i}{2}\hbar\sigma^a\sigma_p\left[J_a^-(z)g(x) + g(x)J_a^+(z)\right]\delta(z-x)\\ &- \tfrac{i}{2}\hbar\sigma_p\left[g(x)\bar{J}_a^-(x) + \bar{J}_a^+(x)g(x)\right]\sigma^a\delta(z-x)\\ =&\tfrac{i}{2}\hbar(2\kappa+\hbar\mathcal{G})\sigma_p g(x)\delta'(z-x) - \tfrac{i}{2}\hbar\beta\sigma_p g'(x)\delta(z-x).\end{aligned} \qquad (B.5)$$

Therefore eqs. (B.4) are satisfied only if

$$\beta = 2\kappa + \hbar\mathcal{G}. \tag{B.6}$$

That is, if we define $g(x)$ with β given by eqn. (B.6) we find that eqs. (B.4) are valid. If we select an alternative value of β to that given by the above equation, the solution to eqn. (B.5) will be non-local (except in the abelian case). Similarly, we can derive eqs. (3.10) regarding the normal ordered $g^{-1}(x)$ operator.

Finally, we derive eqn. (3.12) for $[T(z), g(x)] + [g(z), T(x)]$. Consider

$$\begin{aligned}
[t(z), g(x)] =& \eta^{ab}[J_a^+(z)J_b^+(z) + J_a^-(z)J_b^-(z) + 2J_a^-(z)J_b^+(z), g(x)] \\
=& \eta^{ab}\Big\{2[J_a^+(z), g(x)]J_b^+(z) + 2J_a^-(z)[J_b^-(z), g(x)] \\
&+ 2J_a^-(z)[J_b^+(z), g(x)] + 2[J_a^-(z), g(x)]J_b^+(z) \\
&+ \left[J_a^+(z), [J_b^+(z), g(x)]\right] - \left[J_a^-(z), [J_b^-(z), g(x)]\right]\Big\} \\
=& - i\hbar \left[\sigma^a g(x)J_a^+(x) + \sigma^a J_a^+(x)g(x)\right]\delta(z-x) \\
&- \frac{\hbar^2 c}{4} g(x)\left\{[\delta^+(z-x)]^2 - [\delta^-(z-x)]^2\right\},
\end{aligned} \tag{B.7}$$

where c is defined by eqn. (3.13). Thus

$$[T(z), g(x)] + [g(z), T(x)] = -\frac{i\hbar^2 c}{8\pi(2\kappa + \hbar\mathcal{G})}\Big[2g(x)\delta'(z-x) + g'(x)\delta(z-x)\Big]. \tag{B.8}$$

The corresponding result for $g^{-1}(x)$ (eqn. (3.14)) follows similarly.

Appendix C: Definition and Form of σ_a and σ^a.

It will be convenient to define the lower-case base-elements of the centre G_c of G_o by the equation

$$[\sigma_a, M_\pm] = \pm M_\pm^{(a)} \tag{C.1}$$

The explicit form of these matrices is then given by

$$\sigma_a = I_1 + ... + I_\alpha - \frac{(n_1 + ... + n_\alpha)}{N}\mathbf{I}, \quad (\alpha = a). \tag{C.2}$$

Here $\mathbf{I}$ is the $sl(N)$ identity matrix, and I_α is the restriction of $\mathbf{I}$ to the $sl(n_\alpha)$ block, so that $\mathbf{I} = \sum_\alpha I_\alpha$. The dual matrices σ^a in the centre of G_o are then

$$\sigma^a = \frac{I_\alpha}{n_\alpha} - \frac{I_{\alpha+1}}{n_{\alpha+1}}, \quad (\alpha = a). \tag{C.3}$$

It is clear from eqn. (C.2) that we have

$$M_o = \sum_a \sigma_a. \tag{C.4}$$

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